Exercise 26

Solve the boundary-value problem, if possible.

$$y'' + 6y' = 0, \quad y(0) = 1, \quad y(1) = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 6(re^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 6r = 0$$

Solve for r.

$$r(r+6) = 0$$

$$r = \{-6, 0\}$$

Two solutions to the ODE are e^{-6x} and $e^0 = 1$. By the principle of superposition, then,

$$y(x) = C_1 e^{-6x} + C_2.$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 1$$

$$y(1) = C_1 e^{-6} + C_2 = 0$$

Solving this system of equations yields $C_1 = e^6/(e^6 - 1)$ and $C_2 = 1/(1 - e^6)$. Therefore, the solution to the boundary value problem is

$$\begin{aligned} y(x) &= \frac{e^6}{e^6 - 1} e^{-6x} + \frac{1}{1 - e^6} \\ &= \frac{e^{6(1-x)}}{e^6 - 1} + \frac{1}{1 - e^6} \\ &= \frac{e^{6(1-x)} - 1}{e^6 - 1}. \end{aligned}$$

Below is a graph of $y(x)$ versus x.